DIMENSION AND MEASURE OF BAKER-LIKE SKEW-PRODUCTS OF β-TRANSFORMATIONS

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Abstract. We consider a generalisation of the baker's transformation, consisting of a skew-product of contractions and a β-transformation. The Hausdorff dimension and Lebesgue measure of the attractor is calculated for a set of parameters with positive measure. The proofs use a new transverality lemma similar to Solomyak's [11]. This transversality, which is applicable to the considered class of maps holds for a larger set of parameters than Solomyak's transversality.

1. Introduction

In [1], Alexander and Yorke considered fat baker's transformations. These are maps on the square $[0,1) \times [0,1)$, defined by

$$(x,y) \mapsto \begin{cases} (\lambda x, 2y) & \text{if } y < 1/2 \\ (\lambda x + 1 - \lambda, 2y-1) & \text{if } y \geq 1/2 \end{cases},$$

where $\frac{1}{2} < \lambda < 1$ is a parameter, see Figure 1. They showed that the SRB-measure of this map is the product of Lebesgue-measure and (a rescaled version of) the distribution of the corresponding Bernoulli convolution

$$\sum_{k=1}^{\infty} \pm \lambda^k.$$

Together with Erdős' result [3], this implies that if λ is the inverse of a Pisot-number, then the SRB-measure is singular with respect to the Lebesgue measure on $[0,1) \times [0,1)$.

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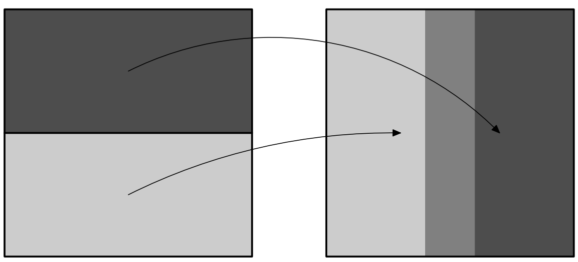

Figure 1. The fat baker's transformation for $\lambda = 0.6$.

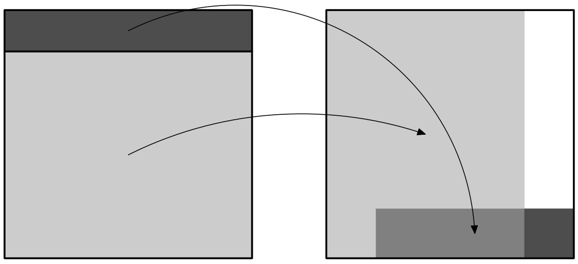

FIGURE 2. The map (2) for $\beta = 1.2$ and $\lambda = 0.8$

In [11], Solomyak proved that for almost all $\lambda \in (\frac{1}{2}, 1)$, the distribution of the corresponding Bernoulli convolution $\sum_{k=1}^{\infty} \pm \lambda^k$ is absolutely continuous with respect to Lebesgue measure. Hence this implies that the SRB-measure of the fat baker's transformation is absolutely continuous for almost all $\lambda \in (\frac{1}{2}, 1)$. Solomyak's proof used a transversality property of power series of the form $g(x) = 1 + \sum_{k=1}^{\infty} a_k x^k$, where $a_k \in \{-1, 0, 1\}$. More precisely, Solomyak proved that there exists a $\delta > 0$ such that if $x \in (0, 0.64)$ then

$$|g(x)| < \delta \quad \Longrightarrow \quad g'(x) < -\delta. \tag{1}$$

This property ensures that if the graph of $g(x)$ intersects the x-axis it does so at an angle which is bounded away from 0, thereby the name transversality. The constant 0.64 is an approximation of a root to a power series and cannot be improved to something larger than this root. A simplified version of Solomyak's proof appeared in the paper [6], by Peres and Solomyak. We will make use of the method from this simpler version.

In this paper we consider maps of the form

$$(x, y) \mapsto \begin{cases} (\lambda x, \beta y) & \text{if } y < 1/\beta \\ (\lambda x + 1 - \lambda, \beta y - 1) & \text{if } y \geq 1/\beta \end{cases}, \tag{2}$$

where $0 < \lambda < 1$ and $1 < \beta < 2$, see Figure 2. Using the above mentioned transversality of Solomyak one can prove that for almost all $\lambda \in (0, 0.64)$ and $\beta \in (1, 2)$ the SRB-measure is absolutely continuous with respect to Lebesgue measure provided $\lambda\beta > 1$, and the Hausdorff dimension of the SRB-measure is $1 + \frac{\log \beta}{\log 1/\lambda}$ provided $\lambda\beta < 1$.

A problem with this approach is that the condition $\lambda < 0.64$ is very restrictive when β is close to 1. Then the above method yields no λ for which the SRB-measure is absolutely continuous, and it does not give the dimension of the SRB-measure for any $\lambda \in (0.64, 1/\beta)$.

We prove that these results about absolute continuity and dimension of the SRB-measure hold for sets of (β, λ) of positive Lebesgue measure, even when $\lambda > 0.64$. This is done by extending the interval on which the transversality property (1) holds. This can be done in our setting, since in our class of maps, not every sequence $(a_k)_{k=1}^{\infty}$ with $a_k \in \{-1, 0, 1\}$ occurs in the power series $g(x) = 1 + \sum_{k=1}^{\infty} a_k x^k$ that we need to consider in the proof. To control which sequences that occur, we will use some results of Brown and Yin [2] and Kwon [4] on natural extensions of β-shifts.

The paper is organised as follows. In Section 2 we recall some facts about β-transformations and β-shifts. We then present the results of Brown and

Yin, and Kwon in Section 3. In Section 4 we state our results, and give the proofs in Section 6. The transversality property is stated and proved in Section 5.

2. β-shifts

Let $\beta > 1$ and define $f_\beta \colon [0,1] \to [0,1)$ by $f_\beta(x) = \beta x$ modulo 1. For $x \in [0,1]$ we associate a sequence $d(x,\beta) = (d_k(x,\beta))_{k=1}^\infty$ defined by $d_k(x,\beta) = [\beta f_\beta^{k-1}(x)]$ where $[x]$ denotes the integer part of x. If $x \in [0,1]$, then $x = \phi_\beta(d(x,\beta))$, where

$$\phi_\beta(i_1, i_2, \dots) = \sum_{k=1}^\infty \frac{i_k}{\beta^k}$$

This representation, among others, of real numbers was studied by Rényi [8]. He proved that there is a unique probability measure μ_β on $[0,1]$ invariant under f_β and equivalent to Lebesgue measure. We will use this measure in Section 6.

We let S_β^+ denote the closure in the product topology of the set $\{\, d(x,\beta) : x \in [0,1) \,\}$. The compact symbolic space S_β^+ together with the left shift σ is called a β-shift. If we define $d_-(1,\beta)$ to be the limit in the product topology of $d(x,\beta)$ as x approaches 1 from the left, we have the equality

$$S_\beta^+ = \{\, (a_1, a_2, \dots) \in \{0, 1, \dots, [\beta]\}^{\mathbb{N}} : \\ \sigma^k(a_1, a_2, \dots) \leq d_-(1,\beta) \ \forall k \geq 0 \,\}, \tag{3}$$

where σ is the left-shift. This was proved by Parry in [5], where he studied the β-shifts and their invariant measures. Note that $d_-(1,\beta) = d(1,\beta)$ if and only if $d(1,\beta)$ contains infinitely many non-zero digits. A particularly useful property of the β-shift is that $\beta < \beta'$ implies $S_\beta^+ \subset S_{\beta'}^+$. The map $\phi_\beta \colon S_\beta^+ \to [0,1]$ is not necessarily injective, but we have $d(\cdot,\beta) \circ f_\beta = \sigma \circ d(\cdot,\beta)$.

3. Symmetric β-shifts

Let $\beta > 1$ and consider S_β^+. The natural extension of (S_β^+, σ) can be realised as (S_β, σ), with

$$S_\beta = \{\, (\dots, a_{-1}, a_0, a_1, \dots) : (a_n, a_{n+1}, \dots) \in S_\beta^+ \ \forall n \in \mathbb{Z} \,\},$$

where σ is the left shift on bi-infinite sequences. We will use the concept of cylinder sets only in S_β. A cylinder set is a subset of S_β of the form

$$[a_{-n}, a_{-n+1}, \dots, a_0] = \{\, (\dots, b_{-1}, b_0, b_1, \dots) \in S_\beta : a_k = b_k \ \forall k = -n, \dots, 0 \,\}.$$

We define S_β^- to be the set

$$\begin{aligned} S_\beta^- &= \{\, (b_1, b_2, \dots) : \exists (a_1, a_2, \dots) \in S_\beta^+ \text{ s.t. } (\dots, b_2, b_1, a_1, a_2, \dots) \in S_\beta \,\} \\ &= \{\, (b_1, b_2, \dots) : (\dots, b_2, b_1, 0, 0, \dots) \in S_\beta \,\}. \end{aligned}$$

We will be interested in the set S of β for which $S_\beta^+ = S_\beta^-$. This set was considered by Brown and Yin in [2]. We now describe the properties of S that we will use later on.

Consider a sequence of the digits a and b. Any such sequence can be written in the form

$$(a^{n_1}, b, a^{n_2}, b, \ldots),$$

where each n_k is a non-negative integer or ∞. We say that such a sequence is allowable if $a \in \mathbb{N}$, $b = a - 1$, and $n_1 \geq 1$. If the sequence $(n_1, n_2, \ldots)$ is also allowable, we say that $(a^{n_1}, b, a^{n_2}, b, \ldots)$ is derivable, and we call $(n_1, n_2, \ldots)$ the derived sequence of $(a^{n_1}, b, a^{n_2}, b, \ldots)$. For some sequences, this operation can be carried out over and over again, generating derived sequences out of derived sequences. We have the following theorem.

Theorem 1 (Brown–Yin [2], Kwon [4])**.** *$\beta \in S$ if and only if $d(1, \beta)$ is derivable infinitely many times.*

The "only if"-part was proved by Brown and Yin in [2] and the "if"-part was proved by Kwon in [4]. Using this characterisation of S, Brown and Yin proved that S has the cardinality of the continuum, but its Hausdorff dimension is zero.

There is a connection between numbers in S and Sturmian sequences. We will not make any use of the connection in this paper, but refer the interested reader to Kwon's paper [4] for details.

For our main results in the next section, it is nice to know whether S contains numbers arbirarily close to 1. The following proposition is easily proved using Theorem 1.

Proposition 1. $\inf S = 1$.

Proof. We prove this statement by explicitely choosing sequences $d(1, \beta)$ corresponding to numbers $\beta \in S$ arbitrarily close to 1. We do this by first finding some sequences that are infinitely derivable, and then we find the corresponding β by solving the equation $1 = \phi_\beta(d(1, \beta))$. Let us first remark that the sequence $(1, 0, 0, \ldots)$ is its own derived sequence.

The sequence $d(1, \beta) = (1, 1, 0, (1, 0)^\infty)$ is clearly derivable infinitely many times. It's derived sequence is $(2, 1, 1, \ldots)$, and the derived sequence of this sequence is $(1, 0, 0, \ldots)$. One finds numerically that the corresponding β is given by $\beta = 1.801938\ldots$ and that $1/\beta = 0.554958\ldots$

There are however smaller numbers in the set S. Consider the sequence $d(1, \beta) = (1, 0, (1, 0, 0)^\infty)$. It's derived sequence is $(1, 1, 0, (1, 0)^\infty))$, which derives to $(2, 1, 1, \ldots)$, and so on. Solving for β we find that $\beta = 1.558980\ldots$ and $1/\beta = 0.641445\ldots$ Now, for all natural n, let β_n be such that

$$d(1, \beta_n) = (1, 0^n, (1, 0^{n+1})^\infty).$$

Then, for $n \geq 2$, the derived sequence of $d(1, \beta_n)$ is the sequence $d(1, \beta_{n-1})$. Hence all sequences $d(1, \beta_n)$ are infinitely derivable, and so $\beta_n \in S$. Moreover it is clear that $\beta_n \to 1$ as $n \to \infty$. See Table 1. $\square$

4. Results

Let $0 < \lambda < 1$ and $1 < \beta < 2$. Put $Q = [0, 1) \times [0, 1)$ and define $T_{\beta,\lambda} \colon Q \to Q$ by

$$T_{\beta,\lambda}(x, y) = \begin{cases} (\lambda x, \beta y) & \text{if } y < 1/\beta \\ (\lambda x + 1 - \lambda, \beta y - 1) & \text{if } y \geq 1/\beta \end{cases}.$$

n	β_n	$1/\beta_n$
1	1.558980...	0.641445...
2	1.438417...	0.695209...
3	1.365039...	0.732580...
4	1.315114...	0.760390...
5	1.278665...	0.782066...

TABLE 1. Some numerical values of β_n.

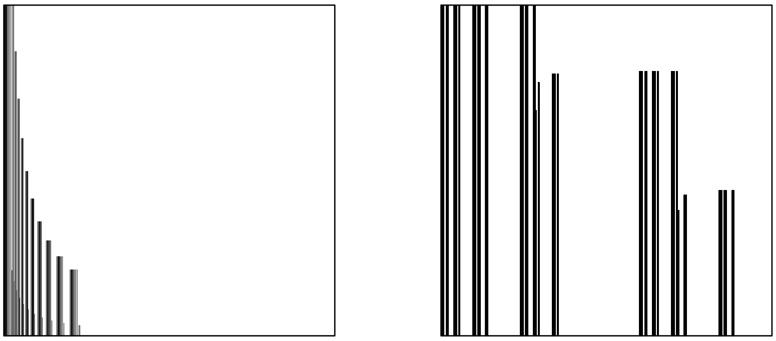

FIGURE 3. The set Λ for $\beta = 1.2$ and $\lambda = 0.8$ (left) and $\beta = 1.8$ and $\lambda = 0.4$ (right).

Denote by ν the 2-dimensional Lebesgue measure on Q. For any $n \in \mathbb{N}$ we define the measure

$$\nu_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu \circ T_{\beta,\lambda}^{-n}.$$

The SRB-measure (it is unique as noted below) of $T_{\beta,\lambda}$ is the weak limit of ν_n as $n \to \infty$.

The SRB-measures are characterised by the property that their conditional measures along unstable manifolds are equivalent to Lebesgue measure. The existence of such measures was established for invertible maps by Pesin [7] and extended to non-invertible maps by Schmeling and Troubetzkoy [10]. We denote the SRB-measure of $T_{\beta,\lambda}$ by μ_{SRB}. Using the Hopf-argument used by Sataev in [9] one proves that the SRB-measure is unique. (Sataev's paper is about a somewhat different map, but the argument goes through without changes.)

The support of μ_{SRB} is the set

$$\Lambda = \text{closure} \bigcap_{n=0}^{\infty} T_{\beta,\lambda}^{n}(Q)$$

of which we have examples in Figure 3. One can estimate the dimension from above by covering the set Λ with the natural covers, consisting of the pieces of $T_{\beta,\lambda}^{n}(Q)$. This gives us the upper bound, that the Hausdorff dimension of Λ is at most $1 + \frac{\log \beta}{\log 1/\lambda}$. If $\lambda\beta > 1$ this is a trivial estimate, since then $1 + \frac{\log \beta}{\log 1/\lambda} > 2$.

The following theorem states that in the case when $\lambda\beta < 1$, there is a set of parameters of positive Lebesgue measure for which the estimate above is optimal.

Theorem 2. *Let* $1 < \beta < 2$ *and* $\gamma = \inf\{\, \beta' \in S : \beta' \geq \beta \,\}$*. Then for Lebesgue almost every* $\lambda \in (0, 1/\gamma)$ *the Hausdorff dimension of the* SRB*-measure of* $T_{\beta,\lambda}$ *is* $1 + \frac{\log \beta}{\log 1/\lambda}$*.*

Recall from Proposition 1 that $\inf S = 1$. This implies that when β gets close to 1, Theorem 2 gives the dimension of the SRB-measure for a large set of $\lambda > 0.64$, which is not obtainable using Solomyak's transversality from [11], described in the introduction.

In the area-expanding case, when $\lambda\beta > 1$, we have the following theorem.

Theorem 3. *For any* $\gamma \in S$*, there is an* $\varepsilon > 0$ *such that for all* β *with* $1/\beta \in [1/\gamma, 1/\gamma + \varepsilon)$*, and Lebesgue almost every* $\lambda \in (1/\beta, 1/\gamma + \varepsilon)$ *the* SRB*-measure of* $T_{\beta,\lambda}$ *is absolutely continuous with respect to Lebesgue measure.*

Since $\inf S = 1$ by Proposition 1, there are β arbitrarily close to 1 for which we have a set of λ of positive Lebesgue measure, where the SRB-measure is absolutely continuous. In particular, this means that for these parameters, the set Λ has positive 2-dimensional Lebesgue measure.

Let us comment on the relation between Theorem 3 and the results of Brown and Yin in [2]. Brown and Yin considers any $\beta > 1$. In the case $1 < \beta < 2$ their result is the following. They consider the map

$$(x, y) \mapsto \begin{cases} (\frac{1}{\beta}x, \beta y) & \text{if } y < \frac{1}{\beta}, \\ (\frac{1}{\beta}x + \frac{1}{\beta}, \beta y - 1) & \text{if } y \geq \frac{1}{\beta}. \end{cases}$$

Hence their map is similar to ours when $\lambda = \frac{1}{\beta}$. They proved that the Lebesgue measure restricted to the set Λ is invariant if $\beta \in S$.

5. Transversality

The main results of this paper, Theorem 2 and Theorem 3, only deal with $1 < \beta < 2$. However, the arguments in this section work just as well for larger β, so for the rest of this section we will be working with a fixed $\beta > 1$.

Consider the set of power series of the form

$$g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k) x^k, \tag{4}$$

where $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ are sequences in S_β^+.

Lemma 1. *There exist* $\varepsilon > 0$ *and* $\delta > 0$ *such that for any power series* g *of the form* (4)*,* $x \in [0, 1/\beta + \varepsilon]$ *and* $|g(x)| < \delta$ *implies that* $g'(x) < -\delta$*.*

Proof. Let

$$0 < \varepsilon < \min\left\{ \frac{1 - 1/\beta}{2}, \frac{1}{[\beta]} \right\} \tag{5}$$

and assume that no such δ exists. We will show that if ε is too small, then we get a contradiction.

By assumption, there is a sequence g_n of power series of the form (4) and a sequence of numbers $x_n \in [0, 1/\beta + \varepsilon]$, such that $\lim_{n\to\infty} g_n(x_n) = 0$ and $\liminf_{n\to\infty} g_n'(x_n) \geq 0$. We can take a subsequence such that g_n converges term-wise to a series

$$g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k) x^k$$

with $(a_1, a_2, \dots), (b_1, b_2, \dots) \in S_\beta^+$, and such that x_n converges to some number $x_0 \in [0, 1/\beta + \varepsilon]$. Clearly, $g(x_0) = 0$ and $g'(x_0) \geq 0$, so looking at (4) we note that $x_0 \neq 0$.

Assume first that $x_0 \in (0, 1/\beta]$. Let $\beta_0 = 1/x_0 \geq \beta$. Then $g(x_0) = 0$ and $(a_1, a_2, \dots), (b_1, b_2, \dots) \in S_{\beta_0}^+$ implies that

$$\phi_{\beta_0}(a_1, a_2, \dots) - \phi_{\beta_0}(b_1, b_2, \dots) = \sum_{k=1}^{\infty} \frac{a_k}{\beta_0^k} - \sum_{k=1}^{\infty} \frac{b_k}{\beta_0^k} = -1. \tag{6}$$

Both of the sums in (6) are in $[0, 1]$, since they equal $\phi_{\beta_0}(a_1, a_2, \dots)$ and $\phi_{\beta_0}(b_1, b_2, \dots)$ respectively. We conclude that

$$\sum_{k=1}^{\infty} \frac{a_k}{\beta_0^k} = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{b_k}{\beta_0^k} = 1.$$

We must therefore have $(a_1, a_2, \ldots) = (0, 0, \ldots)$, and b_k must be nonzero for at least some k. From (4) we then get $g'(x) = -\sum_{k=1}^{\infty} k b_k x^{k-1} < 0$ for all $x \in (0, 1/\beta]$, contradicting the fact that $g'(x_0) \geq 0$.

Assume instead that $x_0 \in (1/\beta, 1/\beta + \varepsilon]$. We write

$$g(x) = 1 + h_1(x) - h_2(x), \tag{7}$$

where

$$h_1(x) = \sum_{k=1}^{\infty} a_k x^k \quad \text{and} \quad h_2(x) = \sum_{k=1}^{\infty} b_k x^k. \tag{8}$$

Since $(b_1, b_2, \dots) \in S_\beta^+$, we have $h_2(1/\beta) \leq 1$. Moreover, for $x \geq 0$ we have $0 \leq h_2'(x) \leq \sum_{k=1}^{\infty} [\beta] k x^{k-1} = \frac{[\beta]}{(1-x)^2}$. Therefore we have

$$h_2(x_0) \leq 1 + \int_{1/\beta}^{1/\beta+\varepsilon} \frac{[\beta]}{(1-x)^2} \mathrm{d}x = 1 + \frac{[\beta]\varepsilon}{(1 - 1/\beta - \varepsilon)(1 - 1/\beta)}. \tag{9}$$

Since $g(x_0) = 0$ we see from (7) and (9) that

$$h_1(x_0) \leq \frac{[\beta]\varepsilon}{(1 - 1/\beta - \varepsilon)(1 - 1/\beta)}.$$

If we have $\frac{[\beta]\varepsilon}{(1-1/\beta-\varepsilon)(1-1/\beta)} \geq x_0$, then let $k = 0$. Otherwise, let k be the largest integer such that $x_0^k > \frac{[\beta]\varepsilon}{(1-1/\beta-\varepsilon)(1-1/\beta)}$. Since $h_1(x)$ is of the form (8) and all its terms are non-negative we must have $a_i = 0$ for $i \leq k$. This implies that

$$h_1'(x) \leq \sum_{i=k+1}^{\infty} [\beta] i x^{i-1} \leq [\beta] \frac{(k+1)x^k + kx^{k+1}}{(1-x)^2} = x^{k+1} [\beta] \frac{k+1+kx}{x(1-x)^2}. \tag{10}$$

By the maximality of k, we have $x_0^{k+1} \leq \frac{[\beta]\varepsilon}{(1-1/\beta-\varepsilon)(1-1/\beta)}$, so (10) and (5) implies

$$h_1'(x_0) \leq \frac{[\beta]^2\varepsilon}{(1-1/\beta-\varepsilon)(1-1/\beta)} \frac{k+1+kx_0}{x_0(1-x_0)^2} \leq \frac{[\beta]^2\varepsilon(2k+1)}{(1-1/\beta-\varepsilon)^4 x_0}. \tag{11}$$

To estimate $h_2'(x_0)$ from below, we note that since $h_2(x)$ is of the form (8), we must have $h_2''(x) \geq 0$ for all x. We also have $h_2(x_0) \geq 1$ since $0 = g(x_0) = h_1(x_0) - h_2(x_0)$. Since $h_2(0) = 0$, this implies

$$h_2'(x_0) \geq \frac{h_2(x_0)}{x_0} \geq \frac{1}{x_0}. \tag{12}$$

Now, if we can choose ε so small that $g'(x_0) = h_1'(x_0) - h_2'(x_0) < 0$, we get a contradiction to the fact that $g'(x_0) \geq 0$. By (11) and (12) we see that it is enough to choose ε so small that

$$\frac{[\beta]^2\varepsilon(2k+1)}{(1-1/\beta-\varepsilon)^4 x_0} - \frac{1}{x_0} < 0 \quad \Longleftrightarrow \quad \varepsilon < \frac{(1-1/\beta-\varepsilon)^4}{[\beta]^2(2k+1)}.$$

So, by (5) it is sufficient to choose

$$\varepsilon < \frac{(1-1/\beta)^4}{2^4[\beta]^2(2k+1)}. \tag{13}$$

To get a bound on k recall that by definition, either $k=0$ or it satisfies

$$x_0^k > \frac{[\beta]\varepsilon}{(1-1/\beta-\varepsilon)(1-1/\beta)}.$$

By (5) we get

$$\begin{aligned} k &< \frac{\log([\beta]\varepsilon) - \log(1-1/\beta-\varepsilon) - \log(1-1/\beta)}{\log(x_0)} \\ &< \frac{\log([\beta]\varepsilon)}{\log(1/\beta+\varepsilon)} \leq \frac{\log([\beta]\varepsilon)}{\log(\frac{1+1/\beta}{2})}. \end{aligned}$$

Inserting this estimate into (13), we get the sufficient condition

$$\varepsilon < \frac{\left(1-1/\beta\right)^4}{2^4[\beta]^2 \frac{2\log([\beta]\varepsilon)}{\log\frac{1+1/\beta}{2}} + 2^4[\beta]^2} \Leftrightarrow \frac{2^5[\beta]^2}{\log\frac{1+1/\beta}{2}}\varepsilon\log([\beta]\varepsilon) + 2^4[\beta]^2\varepsilon < \Big(1-1/\beta\Big)^4. \tag{14}$$

But $\varepsilon\log\varepsilon \to 0$ as ε shrinks to 0, so it is clear that we can find an $\varepsilon > 0$ satisfing (14). □

Remark 1. *Let us give an explicit formula for which ε we can choose in the case $1 < \beta < 2$. For such β we have $[\beta] = 1$. By (5) we have $\varepsilon \leq \frac{1-1/\beta}{2}$, so it follows that $\varepsilon \leq \frac{-\varepsilon\log\varepsilon}{\log\frac{2}{1-1/\beta}}$. This implies that (14) is satisfied if*

$$-\varepsilon\log\varepsilon\Big(\frac{2^5}{\log\frac{2}{1+1/\beta}} + \frac{2^4}{\log\frac{2}{1-1/\beta}}\Big) < \big(1-1/\beta\big)^4.$$

Finally we use that $-\varepsilon \log \varepsilon < \frac{3}{4}\sqrt{\varepsilon}$ *and conclude that it is sufficient to pick any*

$$\varepsilon \leq \frac{16}{9} \frac{\bigl(1-1/\beta\bigr)^8}{\Bigl(\frac{2^5}{\log \frac{2}{1+1/\beta}} + \frac{2^4}{\log \frac{2}{1-1/\beta}}\Bigr)^2}.$$

6. Proofs

Before we give the proofs of Theorems 2 and 3, we make some preparations that will be used in both proofs.

For fixed $1 < \beta < 2$ and $0 < \lambda < 1$, the set Λ satisfies

$$\Lambda = \{ (x,y) : \exists \boldsymbol{a} \in S_\beta \text{ such that } x = \pi_1(\boldsymbol{a}, \lambda),\ y = \pi_2(\boldsymbol{a}, \beta) \}, \tag{15}$$

where

$$\pi_1(\boldsymbol{a}, \lambda) = (1-\lambda) \sum_{k=0}^{\infty} a_{-k} \lambda^k,$$

$$\pi_2(\boldsymbol{a}, \beta) = \sum_{k=1}^{\infty} a_k \beta^{-k}.$$

To see this one can argue as follows. Recall that Λ is the closure of the set $\bigcap_{n=0}^{\infty} T_{\beta,\lambda}^n(Q)$. For each $(x,y) \in \bigcap_{n=0}^{\infty} T_{\beta,\lambda}^n(Q)$, we have that $(x,y) = T_{\beta,\lambda}^n(x_n, y_n)$ for some sequence $(x_n, y_n) \in Q$ with $T_{\beta,\lambda}(x_{n+1}, y_{n+1}) = (x_n, y_n)$. This means that there is a sequence $\boldsymbol{a} \in S_\beta$ such that

$$(x,y) = T_{\beta,\lambda}^n(x_n, y_n) = \biggl(\lambda^n x_n + (1-\lambda) \sum_{k=0}^{n-1} a_{-k} \lambda^k,\ y\biggr),$$

and

$$T_{\beta,\lambda}^n(x,y) = (x_{-n}, y_{-n}) = \biggl(x_{-n},\ \beta^n y - \sum_{k=1}^{n} \beta^{n-k} a_k\biggr).$$

Hence

$$x = \lambda^n x_n + (1-\lambda) \sum_{k=0}^{n-1} a_{-k} \lambda^k,$$

$$y = \beta^{-n} y_{-n} + \sum_{k=1}^{n} \beta^{-k} a_k.$$

Letting $n \to \infty$ we get that all points $(x,y) \in \bigcap_{n=0}^{\infty} T_{\beta,\lambda}^n(Q)$ are of the form $(\pi_1(\boldsymbol{a}, \lambda), \pi_2(\boldsymbol{a}, \beta))$.

For any point $(x,y) \in \Lambda$, there is sequence $(x^{(k)}, y^{(k)})$ of points from $\bigcap_{n=0}^{\infty} T_{\beta,\lambda}^n(Q)$ that converges to (x,y). But each of the points $(x^{(k)}, y^{(k)})$ is of the form $(\pi_1(\boldsymbol{a}^{(k)}, \lambda), \pi_2(\boldsymbol{a}^{(k)}, \beta))$ for some $\boldsymbol{a}^{(k)} \in S_\beta$. Since the space S_β is closed we conclude that $(x,y) \in \Lambda$ is also of this form.

On the other hand, $T_{\beta,\lambda}\bigl(\pi_1(\boldsymbol{a}, \lambda), \pi_2(\boldsymbol{a}, \beta)\bigr) = (\pi_1(\sigma\boldsymbol{a}, \lambda), \pi_2(\sigma\boldsymbol{a}, \beta))$, so the set of points of the form $(\pi_1(\boldsymbol{a}, \lambda), \pi_2(\boldsymbol{a}, \beta)$ is contained in Λ. This proves (15).

We are now going to describe the unstable manifolds using the symbolic representation. Let

$$\pi(\boldsymbol{a}, \beta, \lambda) = (\pi_1(\boldsymbol{a}, \lambda), \pi_2(\boldsymbol{a}, \beta)). \tag{16}$$

Consider a sequence $\boldsymbol{a} \in S_\beta$ and the corresponding point $p = \pi(\boldsymbol{a}, \beta, \lambda)$. In the symbolic space, $T_{\beta,\lambda}$ acts as the left-shift, so the local unstable manifold of p corresponds to the set of sequences $\boldsymbol{b}$ such that $a_k = b_k$ for $k \leq 0$.

For $\lambda \leq 1/2$, π is injective on S_β so the local unstable manifold of p is unique. If $\lambda > 1/2$, then π need not be injective on S_β, so the local unstable manifold of p need not be unique. Indeed, when π is not injective there are $\boldsymbol{a} \neq \boldsymbol{b}$ such that $p = \pi(\boldsymbol{a}, \beta, \lambda) = \pi(\boldsymbol{b}, \beta, \lambda)$, giving rise to different unstable manifolds.

Because of the description (3) we have that $\pi(\boldsymbol{b}, \beta, \lambda)$ is in the unstable manifold of $\pi(\boldsymbol{a}, \beta, \lambda)$ if $(b_1, b_2, \ldots) \leq (a_1, a_2, \ldots)$. Hence for the unstable manifold of $\pi(\boldsymbol{a}, \beta, \lambda)$, there is a maximal $\boldsymbol{c}$, with $c_k = a_k$ for all $k \leq 0$, such that $\pi(\boldsymbol{c}, \beta, \lambda)$ is contained in the unstable manifold. For this $\boldsymbol{c}$ we have that the unstable manifold is the set

$$\{\, (x, y) : x = \pi_1(\boldsymbol{a}, \lambda), y \leq \pi_2(\boldsymbol{c}, \beta) \,\},$$

i.e. a vertical line. So, if $\boldsymbol{a}$ is such that $(a_1, a_2 \ldots)$ does not end with a sequence of zeros, then the unstable manifold has positive length. Since Λ is a union of unstable manifolds, we conclude that Λ is the union of line-segments of the form $\{\, (x, y) : x \text{ fixed}, 0 \leq y \leq c \,\}$.

We will be using the symbolic representation of Λ given by (15), so we transfer the measure μ_{SRB} to a measure η on S_β by $\eta = \mu_{\text{SRB}} \circ \pi(\cdot, \beta, \lambda)$. We take a closer look at this measure η before we start the proofs. Recall, from Section 2, the probability measure μ_β on $[0, 1]$ that is invariant under f_β and equivalent to Lebesgue measure. We get a shift-invariant measure on S_β^+ by taking $\mu_\beta \circ \phi_\beta$ and it can be extended in the natural way to a shift-invariant measure η_β on S_β.

Since μ_{SRB} and μ_β are the unique SRB-measures for $T_{\beta,\lambda}$ and f_β respectively, we conclude that μ_β is the projection of μ_{SRB} to the second coordinate. Thus η and η_β coincide on sets of the form $\{\, \boldsymbol{a} : a_k = b_k, k = 1, \ldots, n \,\}$. By invariance η and η_β will coincide. Since η_β does not depend on λ by construction, η does not depend on λ. We now get the following estimates using the relation between η and μ_β.

$$\begin{aligned}
\eta([a_{-n} \ldots a_0]) &= \mu_\beta\Big(\phi_\beta\big(\{\, (x_i)_{i=1}^\infty \in S_\beta^+ : x_1 \ldots x_{n+1} = a_{-n} \ldots a_0 \,\}\big)\Big) \\
&\leq K \text{ diamater } \Big(\phi_\beta\big(\{\, (x_i)_{i=1}^\infty \in S_\beta^+ : x_1 \ldots x_{n+1} = a_{-n} \ldots a_0 \,\}\big)\Big) \\
&\leq K\beta^{-(n+1)}, \qquad (17)
\end{aligned}$$

where $K < \infty$ is a constant. It follows from (17) that for η almost all $\boldsymbol{a} \in S_\beta$, the sequence $(a_1, a_2, \ldots)$ does not end with a sequence of zeros. As already noted, this means that the unstable manifold is a vertical line segment of positive length. Hence for η almost all $\boldsymbol{a}$ the corresponding unstable manifold is of positive length. We will use this fact in the proofs that follow.

Proof of Theorem 2. Let $\beta > 1$ and pick any $\beta' \geq \beta$ such that $\beta' \in S$. For η almost every sequence $\boldsymbol{a}$, the local unstable manifold of $\pi(\boldsymbol{a}, \beta, \lambda)$ corresponding to $\boldsymbol{a}$, contains a vertical line segment of positive length. Note that this length does not depend on λ. Let ω_δ be the set of sequences $\boldsymbol{a}$, such that the corresponding local unstable manifold of $\pi(\boldsymbol{a}, \beta, \lambda)$ has a length of at least $\delta > 0$. Take $\delta > 0$ so that ω_δ has positive η-measure. Then the set $\Omega_\delta = \pi(\omega_\delta, \beta, \lambda)$ has the same positive μ_{SRB}-measure. Consider the restriction of μ_{SRB} to Ω_δ and project this measure to $[0, 1) \times \{0\}$. Let $\mu^{\mathrm{s}}_{\mathrm{SRB}}$ denote this projection.

Take an interval $I = (c, d)$ with $0 < c < d < 1/\beta'$. Let t be a number in $(0, 1)$. We estimate the quantity

$$J(t) = \int_I \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{1}{|x_1 - x_2|^t} \, \mathrm{d}\mu^{\mathrm{s}}_{\mathrm{SRB}}(x_1) \mathrm{d}\mu^{\mathrm{s}}_{\mathrm{SRB}}(x_2) \mathrm{d}\lambda.$$

If this integral converges, then for Lebesgue almost every $\lambda \in I$, the dimension of $\mu^{\mathrm{s}}_{\mathrm{SRB}}$ is at least t, and so the dimension of μ_{SRB} is at least $1 + t$. Writing $J(t)$ as an integral over the symbolic space we have that

$$J(t) = \int_I \int_{\omega_\delta} \int_{\omega_\delta} \frac{1}{|\pi_1(\boldsymbol{a}, \lambda) - \pi_1(\boldsymbol{b}, \lambda)|^t} \, \mathrm{d}\eta(\boldsymbol{a}) \mathrm{d}\eta(\boldsymbol{b}) \mathrm{d}\lambda.$$

Since η does not depend on λ we can change order of integration and write

$$J(t) = \int_{\omega_\delta} \int_{\omega_\delta} \int_I \frac{1}{|\pi_1(\boldsymbol{a}, \lambda) - \pi_1(\boldsymbol{b}, \lambda)|^t} \, \mathrm{d}\lambda \mathrm{d}\eta(\boldsymbol{a}) \mathrm{d}\eta(\boldsymbol{b}).$$

Now, $\boldsymbol{a}, \boldsymbol{b} \in S_\beta \subset S_{\beta'}$, so for $\boldsymbol{a}$ and $\boldsymbol{b}$ with $a_j = b_j$ for $j = -k + 1, \ldots, 0$ and $a_{-k} \neq b_{-k}$, we have

$$|\pi_1(\boldsymbol{a}, \lambda) - \pi_1(\boldsymbol{b}, \lambda)|^t = \lambda^{kt} |\pi_1(\sigma^{-k} \boldsymbol{a}, \lambda) - \pi_1(\sigma^{-k} \boldsymbol{b}, \lambda)|^t = \lambda^{kt} |g(\lambda)|^t,$$

where g is of the form (4). Since $I = [c, d] \subset [0, 1/\beta']$, we can use the transversality from Lemma 1 to conclude that

$$\int_I \frac{d\lambda}{|\pi_1(\boldsymbol{a}, \lambda) - \pi_1(\boldsymbol{b}, \lambda)|^t} \leq c^{-kt} \int_I \frac{d\lambda}{|g(\lambda)|^t} \leq C c^{-kt} \tag{18}$$

for some constant C. We can write $S_\beta \times S_\beta = A \cup B$, where

$$\begin{aligned} A = & \bigcup_{k=1}^{\infty} \bigcup_{[a_{-k+1}, \ldots, a_0]} [0, a_{-k+1}, \ldots, a_0] \times [1, a_{-k+1}, \ldots, a_0] \\ & \cup \bigcup_{k=1}^{\infty} \bigcup_{[a_{-k+1}, \ldots, a_0]} [1, a_{-k+1}, \ldots, a_0] \times [0, a_{-k+1}, \ldots, a_0], \end{aligned}$$

and

$$B = \bigcup_{\boldsymbol{a} \in S_\beta} \{\boldsymbol{a}\} \times \{\boldsymbol{a}\}.$$

Since $\eta(\boldsymbol{a}) = 0$ for all $\boldsymbol{a} \in S_\beta$, we can replace $\omega_\delta \times \omega_\delta$ by A in the estimates, so after using (18) we get

$$\begin{aligned} J(t) &\leq \sum_{k=1}^{\infty} \sum_{[a_{-k+1},\ldots,a_0]} 2Cc^{-kt} \int_{[0,a_{-k+1},\ldots,a_0]} \int_{[1,a_{-k+1},\ldots,a_0]} \mathrm{d}\eta \mathrm{d}\eta \\ &\leq \sum_{k=1}^{\infty} \sum_{[a_{-k+1},\ldots,a_0]} 2CKc^{-kt}\beta^{-k} \int_{[1,a_{-k+1},\ldots,a_0]} \mathrm{d}\eta \\ &\leq 2CK \sum_{k=0}^{\infty} c^{-kt}\beta^{-k}, \end{aligned}$$

by (17) and the fact that η is a probability measure. This series converges provided that $t < \frac{\log \beta}{\log 1/c}$.

We have now proved that for a.e. λ in $I = (c,d)$, the dimension of the SRB-measure is at least $1 + \frac{\log \beta}{\log 1/c}$. To get the result of the theorem, we let $\varepsilon > 0$ and write $I = (0, 1/\beta')$ as a union of intervals $I_n = (c_n, d_n)$ such that $\frac{\log \beta}{\log 1/c_n} > \frac{\log \beta}{\log 1/d_n} - \varepsilon$. Then the dimension is at least $1 + \frac{\log \beta}{\log c_n} \geq 1 + \frac{\log \beta}{\log 1/\lambda} - \varepsilon$ for a.e. $\lambda \in I$. Since ε and β' was arbitrary this proves the theorem. $\square$

Proof of Theorem 3. In [6], Peres and Solomyak gave a simplified proof of Solomyak's result from [11], about the absolute continuity of the Bernoulli convolution $\sum_{k=1}^{\infty} \pm\lambda^k$. The proof that follows uses the method from [6] and we refer to that paper for omitted details.

Let $\gamma \in S$, pick ε according to Lemma 1 and let β be such that $1/\beta \in [1/\gamma, 1/\gamma + \varepsilon)$. Let $\mu_{\mathrm{SRB}}^{\mathrm{s}}$ be the projection of μ_{SRB} to $[0,1] \times \{0\}$. We form

$$\underline{D}(\mu_{\mathrm{SRB}}^{\mathrm{s}}, x) = \liminf_{r \to 0} \frac{\mu_{\mathrm{SRB}}^{\mathrm{s}}(B_r(x))}{2r},$$

where $B_r(x) = (x-r, x+r)$, and note that $\mu_{\mathrm{SRB}}^{\mathrm{s}}$ is absolutely continuous with respect to Lebesgue measure if $\underline{D}(\mu_{\mathrm{SRB}}^{\mathrm{s}}, x) < \infty$ for $\mu_{\mathrm{SRB}}^{\mathrm{s}}$ almost all x. Since we already have absolute continuity in the vertical direction, it would then follow that μ_{SRB} is absolutely continuous with respect to the two-dimensional Lebesgue measure. If

$$S = \int_I \int_{[0,1]} \underline{D}(\mu_{\mathrm{SRB}}^{\mathrm{s}}, x) \mathrm{d}\mu_{\mathrm{SRB}}^{\mathrm{s}}(x) \mathrm{d}\lambda < \infty,$$

for an interval I, then $\mu_{\mathrm{SRB}}^{\mathrm{s}}$ is absolutely continuous for almost all $\lambda \in I$. So if we prove that S is bounded for $I = [c, 1/\gamma + \varepsilon]$, where $c > 1/\beta$ is arbitrary, then we are done.

Let $I = [c, 1/\gamma + \varepsilon]$ for some fixed $c > 1/\beta$. By Fatou's Lemma we get

$$\begin{aligned} S &\leq \liminf_{r \to 0} (2r)^{-1} \int_I \int_{[0,1]} \mu_{\mathrm{SRB}}^{\mathrm{s}}(B_r(x)) \, \mathrm{d}\mu_{\mathrm{SRB}}^{\mathrm{s}}(x) \mathrm{d}\lambda \\ &= \liminf_{r \to 0} (2r)^{-1} \int_I \int_{S_\gamma} \eta(B_r(\boldsymbol{a}, \lambda)) \, \mathrm{d}\eta(\boldsymbol{a}) \mathrm{d}\lambda. \end{aligned}$$

where $B_r(\boldsymbol{a}, \lambda) = \{\, \boldsymbol{b} : |\pi_1(\boldsymbol{a}, \lambda) - \pi_1(\boldsymbol{b}, \lambda)| < r \,\}$. We have

$$\eta(B_r(\boldsymbol{a}, \lambda)) = \int_{S_\gamma} \chi_{\{\boldsymbol{b} \in S_\gamma \colon |\pi_1(\boldsymbol{a},\lambda) - \pi_1(\boldsymbol{b},\lambda)| \leq r\}}(\boldsymbol{a}) \, \mathrm{d}\eta(\boldsymbol{b}),$$

where χ is the characteristic function. Since η is independent of λ, we can change the order of integration and we get

$$S \leq \liminf_{r\to 0}(2r)^{-1}\int_{S_\gamma}\int_{S_\gamma}\mu_{\mathrm{Leb}}\{\lambda\in I\colon |\pi_1(\boldsymbol{a},\lambda)-\pi_1(\boldsymbol{b},\lambda)|\leq r\}\,\mathrm{d}\eta(\boldsymbol{a})\mathrm{d}\eta(\boldsymbol{b}),$$

where μ_{Leb} is the one-dimensional Lebesgue measure. Now, $\boldsymbol{a},\boldsymbol{b}\in S_\gamma$, so for $\boldsymbol{a}$ and $\boldsymbol{b}$ with $a_j=b_j$ for $j=-k+1,\dots,0$ and $a_{-k}\neq b_{-k}$, we have

$$|\pi_1(\boldsymbol{a},\lambda)-\pi_1(\boldsymbol{b},\lambda)| = \lambda^k|\pi_1(\sigma^{-k}\boldsymbol{a},\lambda)-\pi_1(\sigma^{-k}\boldsymbol{b},\lambda)| = \lambda^k|g(\lambda)|,$$

where g is of the form (4). Since $I=[c,1/\gamma+\varepsilon]$ we can use the transversality from Lemma 1 and we get

$$\begin{aligned}\mu_{\mathrm{Leb}}\{\lambda\in I\colon |\pi_1(\boldsymbol{a},\lambda)-\pi_1(\boldsymbol{b},\lambda)|\leq r\} &\leq \mu_{\mathrm{Leb}}\{\lambda\in I : |g(\lambda)|\leq rc^{-k}\}\\ &\leq \tilde{K}rc^{-k},\end{aligned}$$

for some constant $\tilde{K}<\infty$. As in the proof of Theorem 2, we can disregard the set

$$B=\bigcup_{\boldsymbol{a}\in S_\beta}\{\boldsymbol{a}\}\times\{\boldsymbol{a}\}.$$

and after using (17) we get

$$\begin{aligned}S &\leq \liminf_{r\to 0}(2r)^{-1}\sum_{k=1}^{\infty}\sum_{[a_{-k+1},\dots,a_0]}2\tilde{K}rc^{-k}\int_{[0,a_{-k+1},\dots,a_0]}\int_{[1,a_{-k+1},\dots,a_0]}\mathrm{d}\eta\mathrm{d}\eta\\ &\leq \sum_{k=1}^{\infty}\sum_{[a_{-k+1},\dots,a_0]}\tilde{K}Kc^{-k}\beta^{-k}\int_{[1,a_{-k+1},\dots,a_0]}\mathrm{d}\eta\\ &\leq \tilde{K}K\sum_{k=0}^{\infty}(c\beta)^{-k},\end{aligned}$$

which converges since $c\beta>1$. Since $c>1/\beta$ was arbitrary, we are done. $\square$

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